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An Example of Nonunique, Discontinuous Solutions in Fluid Dynamics

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ABSTRACT

Different discontinuous weak solutions may be obtained for shallow water flow over an obstacle from the same initial conditions depending upon the form in which the governing system of nonlinear hyperbolic differential equations are written. We exhibit two such different solutions obtained analytically from two different forms of the system of equations. We show that a Lax–Wendroff type finite difference scheme yields accurate approximations to these discontinuous solutions when it is applied to the corresponding system of equations.

I. INTRODUCTION

The motion of an incompressible, homogeneous, inviscid, and hydrostatic fluid is governed by a system of "shallow water" equations. We consider one-dimensional "shallow water" flow over an isolated obstacle as shown in Fig. 1. The equations may be written as in [7]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \varphi}{\partial x} + g\frac{\partial H}{\partial x} = 0, \tag{1.1}$$

$$\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x}(\varphi u) = 0, \tag{1.2}$$

where x and t denote the space and time coordinates; u and φ denote the horizontal velocity and the depth of the fluid; and H is the height of an obstacle above the flat lower boundary. The parameter g denotes the vertical acceleration due to gravity.

The number of equations in the system is equal to the number of unknowns. We might expect that appropriate initial and boundary conditions would lead to unique

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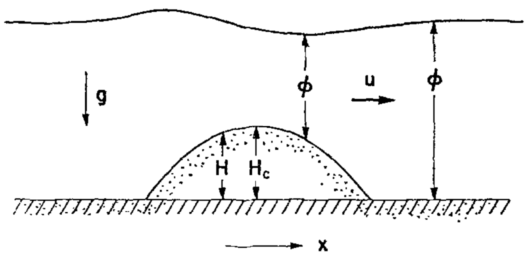

FIG. 1. A cross section view of shallow water flow over an o bstacle.

solutions without recourse to any further physical principles. This expectation, however, is achieved only for the regime of continuous motion. When Eqs. (1.1) and (1.2) are solved for a given initial condition, hydraulic jumps may appear [3]. One purpose of this paper is to illustrate the fact, pointed out by Lax [4], that the "solutions" of such a problem containing shock (or jump) discontinuities are in general not unique. More precisely, Lax showed that discontinuous solutions which arise as weak solutions of a given system of equations are determined by the form in which the equations are written.

A second purpose of this paper is to illustrate concretely the fact (known to many [1]) that a Lax–Wendroff type finite difference scheme will furnish unique accurate approximations to the corresponding unique discontinuous weak solutions. Hence, one may have confidence that a finite difference scheme will adequately describe a discontinuous physical flow for the form in which the differential equations are written.

II. SHOCK CONDITIONS

Let us rewrite the system of equations (1.1) and (1.2) in the velocity form

$$U_t + F_x + B = 0 \tag{2.1}$$

where

$$U = \begin{pmatrix} u \\ \varphi \end{pmatrix}, \qquad F = \begin{pmatrix} u^2/2 + g\varphi \\ u\varphi \end{pmatrix}, \qquad B = \begin{pmatrix} gH_x \\ 0 \end{pmatrix} \tag{2.2}$$

and the subscripts x and t denote differentiation.

By mathematical definition, U is called a *weak solution* of Eq. (2.1) if the integral relation

$$\iint \{W_t U + W_x F - WB\}\, dx\, dt = 0 \tag{2.3}$$

holds for every test vector $W = W(x, t)$ which has continuous first derivatives and which vanishes outside of some bounded region. Eq. (2.3) is obtained formally by multiplying (2.1) by W and applying integration by parts. A weak solution with continuous first derivatives is called a *genuine solution* [4].

Weak solutions need not be differentiable. If U_1 and U_2 are two genuine solutions of (2.1) whose domains in the x, t plane are separated by a smooth curve $\mathscr{L}$, the two taken together will constitute a weak solution if and only if the slope $\tau = dt/dx$ of the separating curve and the value of U_1 and U_2 along the curve satisfy the conditions

$$\frac{1}{\tau}(U_2 - U_1) = F(U_2) - F(U_1), \tag{2.4}$$

valid for each component of the vectors U_i and F_i. (See [4] and also [2], p. 149.) In fact, by integration by parts applied to (2.3), we find

$$\int_{\mathscr{L}} W[(U_2 - U_1)\, dx + (F_1 - F_2)\, dt] = 0$$

for every test function W, whence (2.4) follows. The inverse of τ, denoted by

$$C = \tau^{-1} \tag{2.5}$$

is called the propagation velocity of the discontinuity

Shock Conditions in Velocity Form.

Applying (2.4) and (2.5) to (2.1), the following shock conditions are derived:

$$C = \left(\frac{u_2^2}{2} - \frac{u_1^2}{2} + g\varphi_2 - g\varphi_1\right)\Big/(u_2 - u_1), \tag{2.6}$$

$$C = (u_2\varphi_2 - u_1\varphi_1)/(\varphi_2 - \varphi_1). \tag{2.7}$$

Upon eliminating u_2 from (2.6), we get from (2.7)

$$C = u_1 \pm \left(\frac{2g\varphi_2^2}{\varphi_1 + \varphi_2}\right)^{1/2}. \tag{2.8}$$

The proper choice of sign preceding the radical of (2.8) must be determined.

Condition (2.7) can be derived independently from the requirement for the conservation of mass through the discontinuity and so there is no question about this being correct physically. Condition (2.8), on the other hand, is slightly different from the usual condition for hydraulic jumps which are discussed, for example, by Rouse [6] and Stoker [7].

In order to derive, based on the theory of weak solutions, the same discontinuity conditions as described in [6] and [7], we must first rewrite the system of equations

(1.1) and (1.2) by using as the dependent variables momentum and mass instead of velocity and mass. Let us define the momentum by

$$m = u\varphi. \tag{2.9}$$

Multiplying (1.1) and (1.2) by φ and u, respectively, and adding the two resulting equations, it follows that

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(\frac{m^2}{\varphi}\right) + g\varphi\frac{\partial \varphi}{\partial x} + g\varphi\frac{\partial H}{\partial x} = 0. \tag{2.10}$$

The system of equations (2.10) and (1.2) can be put into the following momentum form

$$V_t + G_x + K = 0, \tag{2.11}$$

where

$$V = \begin{pmatrix} m \\ \varphi \end{pmatrix}, \qquad G = \begin{pmatrix} \dfrac{m^2}{\varphi} + \dfrac{1}{2}g\varphi^2 \\ m \end{pmatrix}, \qquad K = \begin{pmatrix} g\varphi\dfrac{\partial H}{\partial x} \\ 0 \end{pmatrix}. \tag{2.12}$$

Shock Conditions in Momentum Form.

If we replace F by G and U by V in (2.4), the following shock conditions are derived [after using (2.9)] in terms of the original dependent variables u and φ:

$$C = \frac{u_2^2\varphi_2 - u_1^2\varphi_1 + \frac{1}{2}g(\varphi_2^2 - \varphi_1^2)}{\varphi_2 u_2 - \varphi_1 u_1}, \tag{2.13}$$

$$C = \frac{u_2\varphi_2 - u_1\varphi_1}{\varphi_2 - \varphi_1}. \tag{2.14}$$

By using (2.14),.which is identical to (2.7), to eliminate u_2 from (2.13), we obtain

$$C = u_1 \pm \left[g\frac{\varphi_2}{\varphi_1}\left(\frac{\varphi_2 + \varphi_1}{2}\right)\right]^{1/2}. \tag{2.15}$$

Again, the proper choice of sign preceding the radical of (2.15) remains to be made.

Condition (2.15), which is different from (2.8), is well known and has been independently derived from physical considerations (e.g. see [7]). In order to see the effect of using condition (2.8) or (2.15), let us consider a simple jump situation having on one side a depth φ_1 and a velocity u_1 and on the other a depth φ_2 and a velocity u_2, as shown in Fig. 2. For prescribed values of u_1, φ_1, and φ_2, the propagation speed C and velocity u_2 can be determined by solving the system (2.7) and (2.8) or the system (2.14) and (2.15). We choose the plus sign both in (2.8) and (2.15).

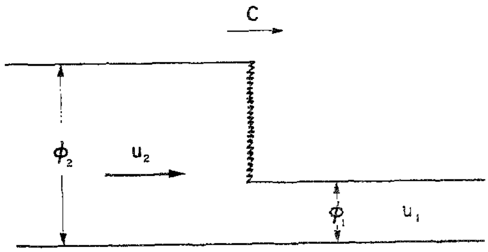

FIG. 2. Simple jump configuration used to compare shock conditions.

a. Jump conditions in velocity form

$$u_2 = u_1 + (r - 1)\left(\frac{2}{1 + r}\right)^{1/2}(g\varphi_1)^{1/2},$$
$$C = u_1 + r\left(\frac{2}{1 + r}\right)^{1/2}(g\varphi_1)^{1/2}, \tag{2.16}$$

where

$$r = \varphi_2/\varphi_1 . \tag{2.17}$$

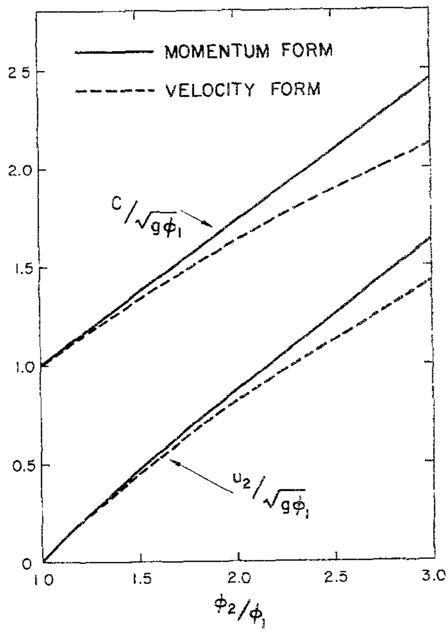

FIG. 3. Dimensionless flow velocity $u_2/(g\varphi_1)^{1/2}$ and jump velocity $C/(g\varphi_1)^{1/2}$ for the simple jump configuration shown in Fig. 2. Momentum form results are denoted by solid lines and velocity form results are denoted by dashed lines.

b. Jump conditions in momentum form

$$u_2 = u_1 + (r-1)\left(\frac{1+r}{2r}\right)^{1/2}(g\varphi_1)^{1/2},$$
$$C = u_1 + r\left(\frac{1+r}{2r}\right)^{1/2}(g\varphi_1)^{1/2}. \tag{2.18}$$

Fig. 3 shows a comparison of cases a and b in which $u_1 = 0$. The abscissa denotes φ_2/φ_1, and the ordinate denotes the dimensionless values of $C/(g\varphi_1)^{1/2}$ and $u_2/(g\varphi_1)^{1/2}$. The solid and dashed lines represent the momentum form (2.18) and velocity form (2.16), respectively. The difference in the values of C and u_2 generated by the two shock conditions, a and b, increases with the ratio φ_2/φ_1.

III. Numerical Solutions

The foregoing simple example clearly shows that the formulation of (weak solution) jump conditions associated with a given system of equations depends on the form in which the equations are written. Now the question arises whether these solutions containing jumps can be found when the same physical problem is solved by applying a finite difference scheme to the different forms of the equations but with the same initial conditions. In order to investigate this question in a fairly complex situation, let us consider the following physical problem related to Fig. 1.

For $t < 0$ and $-\infty < x < \infty$, the fluid is completely at rest, and the height of the free surface, denoted by h_0, is constant. The fluid is impulsively set in motion at $t = 0$ so that for $-\infty < x < \infty$ the fluid has a constant horizontal velocity u_0. The problem is then to determine the subsequent motion of the fluid. This problem was treated both analytically and numerically with the use of the momentum form [i.e., Eqs. (2.11)] by Houghton and Kasahara [3].

In this section, we treat the same physical problem numerically, but use the velocity form [i.e., Eqs. (2.1)]. We then compare the two solutions based on identical initial and boundary conditions and space and time increments. We used a numerical scheme developed by Lax and Wendroff [5] and refer to [3] for a detailed description of the computational procedure. We shall present only results here. Note that we use the initial condition $u_0/(gh_0)^{1/2} = 0.7$. The ratio of the height of the obstacle crest to the initial depth H_c/h_0 was set at 0.5. A total of 2000 spatial grid points are used in the computations.

Fig. 4a shows the numerical solutions of velocity and the height of the free surface after 400 time steps where the velocity form of equations (2.1) has been used. The symbol V in the upper right corner stands for the velocity form. Fig. 4b is the same as Fig. 4a except that the momentum form (2.11) was used. The symbol M in the upper right corner stands for the momentum form. If we compare Fig. 4a with

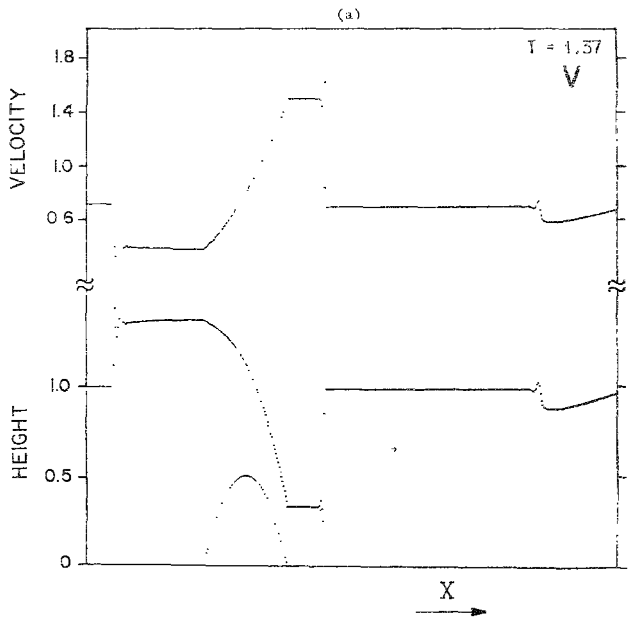

FIG. 4a Numerical solution for the velocity form equations after 400 time steps.

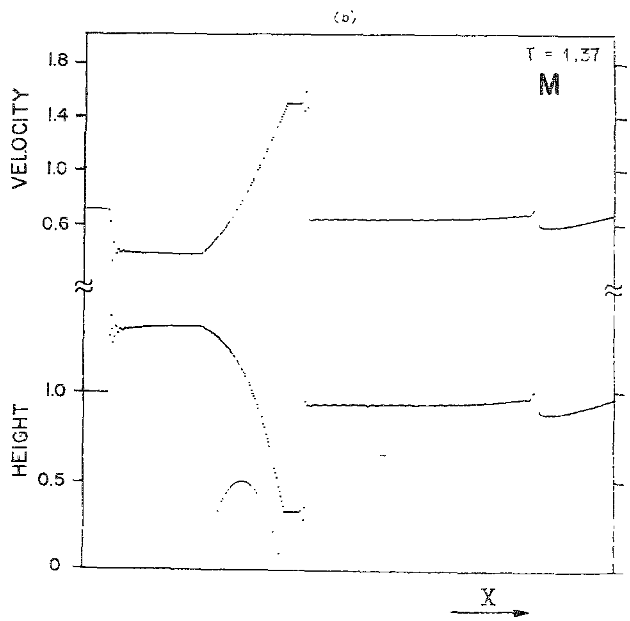

FIG. 4b. Same as Fig. 4a but for the momentum form equations.

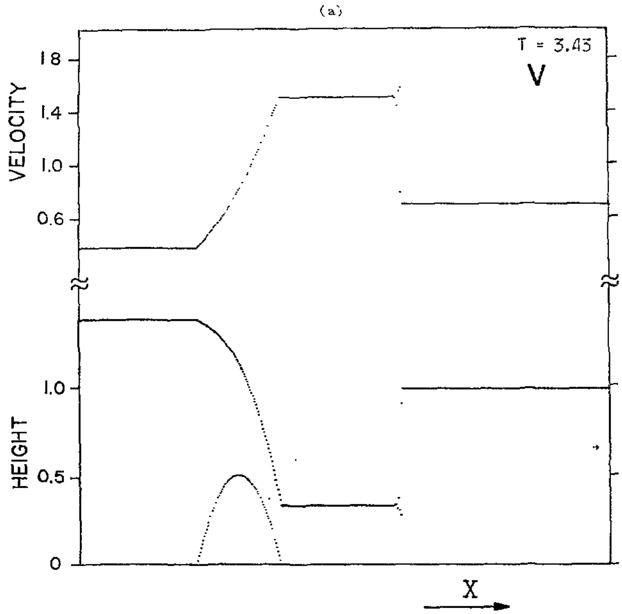

FIG. 5a. Same as Fig. 4a but after 1000 time steps.

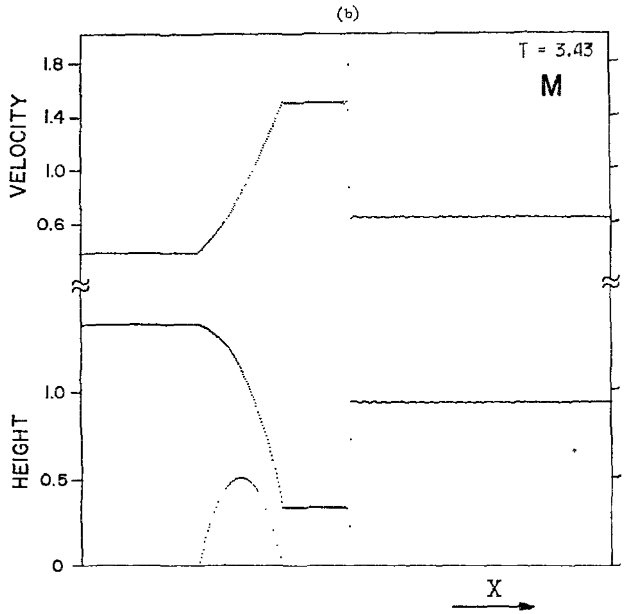

FIG. 5b. Same as Fig. 4b but after 1000 time steps.

Fig. 4b, we note marked differences in the positions of the downstream jump. This difference in position of the downstream jump becomes more pronounced after 1000 time steps, as shown in Figs. 5a and 5b.

IV. Analytical Solutions

As seen from Figs. 4 and 5, the structure of the flow in the neighborhood of the obstacle is independent of time. Houghton and Kasahara [3] have shown that after sufficient time has elapsed, the solution in the neighborhood of the obstacle can be determined by analyzing the steady-state solutions of the relevant equations by taking into account proper jump conditions. In the analytical solutions the discontinuity is regarded as an interior boundary. Jump conditions are applied to piece together analytic steady state solutions for the purpose of obtaining the asymptotic structure of the flow over an isolated obstacle. But, in the numerical scheme, no special effort is needed to compute the jump: it appears naturally in the numerical values of the solution. (See [8] where schemes that have this property are derived from physical considerations.)

The asymptotic solution for the initial depth h_0, the initial velocity u_0, and a smooth obstacle with a crest height of H_c can be determined by solving the following ten algebraic equations for the ten variables shown in Fig. 6. For the momentum form the ten equations are (see [3]):

a. Jump conditions on the upstream side of the obstacle

$$C_l = \frac{h_0 u_0 - h_A u_A}{h_0 - h_A}, \tag{4.1}$$

$$C_l = u_0 - \left(\frac{g h_A}{h_0}\left(\frac{h_A + h_0}{2}\right)\right)^{1/2}. \tag{4.2}$$

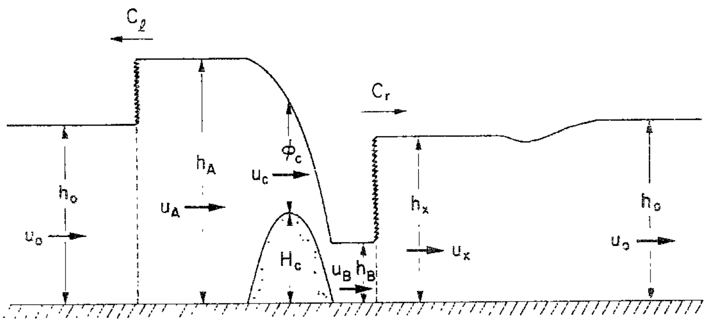

Fig. 6. Asymptotic conditions in the vicinity of the obstacle. u_0 and h_0 are initial conditions. For given H_c, the other ten variables are unknowns determinable by the ten algebraic equations in Section 4.

Equation (4.1) comes from (2.14), and Eq. (4.2) (with a minus sign preceding the radical) comes from (2.15).

b. Steady flow over the obstacle

$$\frac{u_c^2}{2g} + \varphi_c + H_c = \frac{u_A^2}{2g} + h_A , \tag{4.3}$$

$$u_c \varphi_c = u_A h_A , \tag{4.4}$$

$$\frac{u_B^2}{2g} + h_B = \frac{u_A^2}{2g} + h_A , \tag{4.5}$$

$$u_B h_B = u_A h_A . \tag{4.6}$$

c. Critical condition at the crest of the obstacle

$$u_c = (g\varphi_c)^{1/2}. \tag{4.7}$$

d. Jump conditions on the downstream side of the obstacle

$$C_r = \frac{h_B u_B - h_x u_x}{h_B - h_x} , \tag{4.8}$$

$$C_r = u_B - \left(\frac{g h_x}{h_B}\left(\frac{h_x + h_B}{2}\right)\right)^{1/2}. \tag{4.9}$$

e. Rarefaction condition downstream of the obstacle

$$u_x - 2(g h_x)^{1/2} = u_0 - 2(g h_0)^{1/2}. \tag{4.10}$$

The ten equations are the same for the velocity form except that the two jump conditions, (4.2) and (4.9), are replaced, respectively, by

$$C_l = u_0 - \left(2g \frac{h_A^2}{(h_0 + h_A)}\right)^{1/2} \tag{4.11}$$

and

$$C_l = u_B - \left(2g \frac{h_x^2}{(h_x + h_B)}\right)^{1/2}. \tag{4.12}$$

Equations (4.11) and (4.12) are based on (2.8) with the sign preceding the radical taken as minus. These two sets of ten equations were solved by the method described in the Appendix. As in the numerical computations, we set $u_0/(g h_0)^{1/2} = 0.7$ and $H_c/h_0 = 0.5$. The results of analytical solutions are compared with those of numerical solutions in Table I. Values taken from the numerical computations are shown only to the number of significant digits that can be determined from the solutions. Note the good agreement between the numerical and analytical solutions for both the velocity and momentum forms of the equations.

TABLE I

COMPARISON OF ANALYTICAL AND NUMERICAL RESULTS

Asymptotic Quantity	Velocity Form Equations Analytical Value	Velocity Form Equations Computed Value	Momentum Form Equations Analytical Value	Momentum Form Equations Computed Value
h_A/h_0	1.3710	1.3709	1.3677	1.3676
$u_A/(gh_0)^{1/2}$	0.3593	0.3593	0.3579	0.3580
$C_l/(gh_0)^{1/2}$	−0.5592	−0.555	−0.5724	−0.569
φ_c/h_0	0.6237	0.6236	0.6211	0.6211
$u_c/(gh_0)^{1/2}$	0.7897	0.7897	0.7881	0.7883
h_B/h_0	0.3315	0.3314	0.3298	0.3298
$u_B/(gh_0)^{1/2}$	1.4860	1.4860	1.4846	1.4846
$C_r/(gh_0)^{1/2}$	0.2737	0.28	0.1541	0.15
h_x/h_0	0.9827	0.9826	0.9281	0.928
$u_x/(gh_0)^{1/2}$	0.6826	0.6825	0.6268	0.623

V. CONCLUSIONS

The excellent agreement between the analytical and numerical solutions for both the velocity form and the momentum form of the equations clearly indicates that the nonuniqueness of weak solutions has been illustrated in the numerical solutions of the Lax–Wendroff finite difference equations as well as in the analytical solutions of the differential equations.

The conclusions made here also apply to the more general problem of hydrodynamic shocks in the flow of a compressible fluid where it is customary to use the system consisting of conservation of mass, momentum, and energy (see, for example, [1]). The Rankine–Hugoniot shock conditions can be derived for this system if Lax's theory of weak solutions is applied. However, as noted by Courant ([2], p. 490), different shock solutions will be obtained for the same physical problem by using the systems of equations for conservation of mass, momentum, and entropy.

APPENDIX—SOLUTION OF THE TEN ALGEBRAIC EQUATIONS FOR ASYMPTOTIC CONDITIONS

Since Houghton and Kasahara [3] have not discussed how to solve the ten algebraic equations to determine the asymptotic structure of the flow over the obstacle, we explain here a method of solution.

Equations (4.1) and (4.2) were combined to eliminate C_l and written such that for a given value of h_0, u_0 is a function of h_A and u_A. Also (4.4) and (4.7) were used to eliminate u_c and φ_c from (4.3). The result is an expression for H_c as a function of h_A and u_A. Thus, we have two equations for u_0 and H_c as functions of h_A and u_A. Note that u_0 increases monotonically with increasing h_A and u_A, whereas H_c increases monotonically with increasing h_A but decreasing u_A. Therefore, a simple iteration procedure can be used to find the values of h_A and u_A. With the obtained values of u_A and h_A, C_l, φ_c, and u_c can be computed using (4.1), (4.4), and (4.7).

Elimination of u_B between (4.5) and (4.6) gives a cubic equation of h_B for known values of u_A and h_A. This cubic equation can be reduced to a quadratic equation by assuming that $h_B \neq h_A$ since the symmetrical condition is not acceptable. By solving the quadratic equation, we find

$$h_B = \frac{u_A}{4g}\left[u_A + (u_A{}^2 + 8gh_A)^{1/2}\right].$$

The solution of u_B can then be found from (4.6) if h_B is known.

Finally, Eqs. (4.8)–(4.10) are solved as follows. First u_x is eliminated from (4.8) using (4.10). Let us call the resulting equation (A). We then start with a small initial guess for h_x and gradually increase the value of h_x until Eqs. (A) and (4.9) yield sufficiently close values for C_r. Using this solution of h_x, the solution of u_x can be obtained from (4.10).

The other set of ten equations for the velocity form can be solved similarly.

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